Existence of a Market Equilibrium

Suppose there are ℓ goods and let $\mathcal{P} \subseteq \mathbb{R}^\ell_+$ be the set of possible price-lists $\mathbf{p}$.

Definition: A **market excess demand function** (or **net demand function**) is a function $\mathcal{Z} : \mathcal{P} \to \mathbb{R}^\ell$.

Notation: Let S denote the unit simplex in $\mathbb{R}^\ell$, $S = \{\mathbf{p} \in \mathbb{R}^\ell_+ \mid \sum_1^\ell p_k = 1\}$.

We assume that a market excess demand function satisfies the following assumptions:

Assumptions:

(A1) $S \subseteq \mathcal{P}$.

(A2) $\forall \mathbf{p} \in \mathcal{P} : \mathbf{p} \cdot \mathcal{Z}(\mathbf{p}) = 0$. (Walras' Law)

(A3) $\mathcal{Z}$ is continuous.

Note that if $\mathcal{Z}(\cdot)$ is the sum of individual behavioral (demand/supply) functions $\mathcal{Z}^i(\cdot)$, then A2 and A3 will be satisfied if they're satisfied by each of the indidvidual functions. Note too that if each individual function $\mathcal{Z}^i(\cdot)$ is homogeneous of degree zero, then $\sum \mathcal{Z}^i$ will be as well, so that if we restrict ourselves to price-lists in the simplex S, we'll effectively be including all possible price-lists.

An **equilibrium** of a market demand function $\mathcal{Z}(\cdot)$ is a price-list that clears all the markets, as in the following definition.

Definition: An **equilibrium** of the market excess demand function $\mathcal{Z}$ is a price-list $\mathbf{p}$ for which

$$\mathcal{Z}(\mathbf{p}) \leqq \mathbf{0} \qquad \text{and} \qquad p_k > 0 \Rightarrow \mathcal{Z}_k(\mathbf{p}) = 0 \ (k = 1, \ldots, \ell)\,.$$

The following remark says that if $\mathcal{Z}(\cdot)$ satisfies Walras' Law, then a $\mathbf{p}$ that satisfies the first condition in the above definition must satisfy the second condition as well, so a price-list $\mathbf{p}$ that satisfies $\mathcal{Z}(\mathbf{p}) \leqq \mathbf{0}$ is an equilibrium.

Remark: If $\mathcal{Z}(\mathbf{p}) \leqq \mathbf{0}$, and if $\mathcal{Z}$ satisfies Walras' Law, then $\mathbf{p}$ is an equilibrium for $\mathcal{Z}(\cdot)$.

Proof: If $\mathcal{Z}_k(\mathbf{p}) \leqq 0$ for each k, then (since $\mathbf{p} \in \mathbb{R}^\ell_+$) no term in the sum $\mathbf{p} \cdot \mathcal{Z}(\mathbf{p})$ can be positive. According to Walras' Law, the sum is always zero, so at this $\mathbf{p}$ each term must be zero — *i.e.*, the second condition in the definition of equilibrium must be satisfied. $\|$

Here's a clever proof, due to Arrow & Hahn, of the existence of a market equilibrium for the case of just two goods:

Theorem: If $\mathcal{Z} : \mathcal{P} \to \mathbb{R}^\ell$ satisfies (A1) - (A3), then $\mathcal{Z}$ has an equilibrium.

Proof for $\ell = 2$ (Arrow & Hahn):

Define $\widetilde{\mathcal{Z}} : [0,1] \to \mathbb{R}^2$ by $\widetilde{\mathcal{Z}}(p_1) = \mathcal{Z}(p_1, 1-p_1)$. The two component functions of $\widetilde{\mathcal{Z}}$ are therefore $\widetilde{\mathcal{Z}}_1(p_1) = \mathcal{Z}_1(p_1, 1-p_1)$ and $\widetilde{\mathcal{Z}}_2(p_1) = \mathcal{Z}_2(p_1, 1-p_1)$. (See Figure 1.) Assumption A1 ensures that $\widetilde{\mathcal{Z}}$ is well-defined on $[0,1]$. According to the Remark above, if there is a price $p_1 \in [0,1]$ at which $\mathcal{Z}_1(p_1, 1-p_1) \leqq 0$ and $\mathcal{Z}_2(p_1, 1-p_1) \leqq 0$, then the price-list $\mathbf{p} = (p_1, 1-p_1)$ is an equilibrium for $\mathcal{Z}$, and we'll say that p_1 is an equilibrium of $\widetilde{\mathcal{Z}}$.

We first consider the two extreme points of $[0,1]$, namely $p_1 = 0$ and $p_1 = 1$. At $p_1 = 0$, Walras' Law (A2) guarantees that $0\widetilde{\mathcal{Z}}_1(0) + 1\widetilde{\mathcal{Z}}_2(0) = 0$, and thus that $\widetilde{\mathcal{Z}}_2(0) = 0$. Therefore, if $\widetilde{\mathcal{Z}}_1(0) \leqq 0$, that's enough to ensure that $p_1 = 0$ is an equilibrium. The same argument establishes that $\widetilde{\mathcal{Z}}_1(1) = 0$, so that if $\widetilde{\mathcal{Z}}_2(1) \leqq 0$, then $p_1 = 1$ is an equilibrium. In either case, the proof would be complete; therefore we assume that $p_1 \neq 0$ and $p_1 \neq 1$, which implies that $\widetilde{\mathcal{Z}}_1(0) > 0$ and $\widetilde{\mathcal{Z}}_2(1) > 0$, and we'll show that some $p_1 \in (0,1)$ must be an equilibrium.

Since $\widetilde{\mathcal{Z}}_2(1) > 0$ and $\widetilde{\mathcal{Z}}_2(\cdot)$ is continuous, there is an interval $(\xi, 1) \subset [0,1]$ of p_1-values at which $\widetilde{\mathcal{Z}}_2(p_1) > 0$. For each such $p_1 \in (\xi, 1)$, Walras's Law guarantees (since $p_1 > 0$ and $1 - p_1 > 0$) that $\widetilde{\mathcal{Z}}_1(p_1) < 0$. But we also have $\widetilde{\mathcal{Z}}_1(0) > 0$ (because we're assuming that $p_1 = 0$ is not an equilibrium). Now we have $\widetilde{\mathcal{Z}}_1(0) > 0$ and $\widetilde{\mathcal{Z}}_1(\overline{p}_1) < 0$ for any $\overline{p}_1 \in (\xi, 1)$, and $\widetilde{\mathcal{Z}}_1$ is defined and continuous for all $p_1 \in [0, \overline{p}_1]$; therefore the Intermediate Value Theorem ensures that there is a $\widehat{p}_1 \in [0, \overline{p}_1]$ at which $\widetilde{\mathcal{Z}}_1(\widehat{p}_1) = 0$. Using Walras' Law again, and the fact that $\widehat{p}_2 = 1 - \widehat{p}_1 > 0$, we have $\widetilde{\mathcal{Z}}_2(\widehat{p}_1) = 0$. Therefore $\widehat{p}_1$ is an equilibrium of $\widetilde{\mathcal{Z}}$, and $(\widehat{p}_1, 1 - \widehat{p}_1)$ is an equilibrium of $\mathcal{Z}$. $\|$

The above proof depends on using Walras' Law to reduce the problem to a single market, applying the Intermediate Value Theorem to show that that market clears, and then using Walras' Law again to establish that the (only) remaining market must clear as well. With more than two goods, Walras' Law doesn't transform the problem to one where the Intermediate Value Theorem applies.

With more than two goods, the mathematical tool we need is the Brouwer Fixed Point Theorem:

Brouwer Fixed Point Theorem: If S is a nonempty, compact, convex set in $\mathbb{R}^\ell$ and $f : S \to S$ is a continuous function, then f has a fixed point — a point $\mathbf{p} \in S$ such that $f(\mathbf{p}) = \mathbf{p}$.

Equilibrium as a Fixed Point

If S is the set of states of a system, and $f : S \to S$ is a function that describes the transition of the system from state to state — *i.e.*, if $s_{t+1} = f(s_t)$ — then it's clear that we ought to simply define an equilibrium of the system as a fixed point of f. However, the equilibrium concept we've been using for markets (a price-list at which the markets clear) was not derived from such a transition function, so we don't have a function f whose fixed points are the equilibria of our system of markets. This is because we don't know exactly how prices always move over time — we don't know the correct, true function f.

Nevertheless, our definition of equilibrium *was* motivated by an informal and incomplete notion of how prices move: *viz.*, if any of the goods are in excess demand or supply, we expect that one or more of their prices will rise or fall. In other words, if there are any markets that aren't cleared by the current prices, then we assume those prices will change — they're not equilibrium prices. And conversely, we assume that if the current prices *do* clear all the markets, then the prices will not change. Thus, while we don't know the precise transition function (or "laws of motion") of the actual market system, we make some weak assumptions about the properties of the transition function, and we (implicitly) use these assumptions to define equilibrium.

Now, with our definition of equilibrium in hand, we can work backwards to *make up* a transition function for which a fixed point is a state that satisfies our definition of equilibrium — *i.e.*, we have to make up a transition function with the property that the price-list changes if and only if there are some markets that don't clear at the current price-list. Then the equilibria are automatically the fixed points of our transition function. And in order to establish that an equilibrium exists, we have to establish that our transition function has a fixed point. Therefore we need to have one or more theorems that guarantee a function will have a fixed point, and we have to make sure the transition function we make up actually satisfies the assumptions of one of these theorems. In the case of Brouwer's Theorem, we need to make sure that the state space S is nonempty, compact, and convex, and make sure that the transition function we make up is continuous. Alternatively, in some situations we might be able to make up a transition function that's contraction, and then use the Banach Contraction Mapping Theorem. For market equilibrium, it's the Brouwer Theorem that does what we need. (It's also worth pointing out that a transition function whose fixed points are equilibria is often extremely useful for *computing* an equilibrium in particular applications of equilibrium analysis. Note how this is suggested by Figure 2, below: imagine iterating the function f from any starting price-list $\mathbf{p}$.)

Gale-Nikaido Existence Proof

Theorem: If $\mathcal{Z} : \mathcal{P} \to \mathbb{R}^\ell$ satisfies (A1) - (A3), then $\mathcal{Z}$ has an equilibrium.

Proof:

For each $k = 1, \ldots, \ell$ define $M_k : \mathbb{R} \to \mathbb{R}$ by $M_k(\mathbf{p}) = \max\{0, \mathcal{Z}_k(\mathbf{p})\}$. Define a "transition function" $f : S \to S$ by

$$f(\mathbf{p}) = \frac{1}{\sum_1^\ell [p_k + M_k(\mathbf{p})]} \left[\mathbf{p} + M(\mathbf{p})\right], \ \forall \mathbf{p} \in S. \qquad \text{(See Figure 2)}$$

We first show that f has a fixed point, by establishing that

$$f \text{ indeed maps } S \text{ into } S \quad \text{ and } \quad f \text{ is continuous,}$$

and then applying Brouwer's Fixed Point Theorem. Then we will show that any fixed point of f is an equilibrium of $\mathcal{Z}$.

To see that f maps S into S, note that for each $\mathbf{p} \in S$ and for each k we have $M_k(\mathbf{p}) \geqq 0$, and therefore $p_k + M_k(\mathbf{p}) \geqq 0$. Since $p_k > 0$ for some k, we have $p_k + M_k(\mathbf{p}) > 0$ for that k, and therefore $\sum_1^\ell [p_k + M_k(\mathbf{p})] > 0$, which ensures that $f(\mathbf{p})$ is well-defined. We need to also verify that the sum of the components of $f(\mathbf{p})$ satisfies $\sum_1^\ell f_k(\mathbf{p}) = 1$, which is clearly true, as the denominator in the definition of $f(\mathbf{p})$ is the sum of the components of $p + M(\mathbf{p})$. Therefore we've established that $\forall \mathbf{p} \in S : f(\mathbf{p}) \in S$. To see that f is continuous, note that each $M_k(\cdot)$ is continuous, so f is a sum and quotient of continuous functions (with denominator strictly positive for all $\mathbf{p}$). Since the function $f : S \to S$ is continuous, and S is nonempty, compact, and convex, Brouwer's Theorem guarantees that f has at least one fixed point.

Now suppose that $\mathbf{p}$ is a fixed point of f — *i.e.*, $f(\mathbf{p}) = \mathbf{p}$, or equivalently,

$$\mathbf{p} + M(\mathbf{p}) = \sum_{k=1}^{\ell} [p_k + M_k(\mathbf{p})]\, \mathbf{p}\,.$$

Then

$$\mathbf{p} \cdot \mathcal{Z}(\mathbf{p}) + M(\mathbf{p}) \cdot \mathcal{Z}(\mathbf{p}) = \sum_{k=1}^{\ell} [p_k + M_k(\mathbf{p})]\, \mathbf{p} \cdot \mathcal{Z}(\mathbf{p})\,.$$

Walras' Law yields $\mathbf{p} \cdot \mathcal{Z}(\mathbf{p}) = 0$, and therefore $M(\mathbf{p}) \cdot \mathcal{Z}(\mathbf{p}) = 0$ — *i.e.*,

$$M_1(\mathbf{p})\mathcal{Z}_1(\mathbf{p}) + \cdots + M_\ell(\mathbf{p})\mathcal{Z}_\ell(\mathbf{p}) = 0\,.$$

According to the definition of the functions $M_k(\cdot)$, we have each $M_k(\mathbf{p}) \geqq 0$, and if $\mathcal{Z}_k(\mathbf{p}) < 0$ then $M_k(\mathbf{p}) = 0$. Therefore each term in the above sum is nonnegative, and since the sum is zero, each term must be zero. Therefore we can't have $\mathcal{Z}_k(\mathbf{p}) > 0$ for any k, because that would yield $M_k(\mathbf{p})\mathcal{Z}_k(\mathbf{p}) > 0$. Therefore $\mathcal{Z}_k(\mathbf{p}) \leqq 0$ for each k, thereby establishing that $\mathbf{p}$ is an equilibrium of $\mathcal{Z}$. ‖

There's nothing special about the particular price-adjustment functions $M_k(\cdot)$ we used in the proof. Any (continuous!) functions that increase p_k when good k is in excess demand and (weakly) decrease p_k when good k is in excess supply, and that are zero (*i.e.*, p_k doesn't change) when there is no excess demand or supply of good k, will work the same way in the proof. In particular, if we let $\lambda \in \mathbb{R}_{++}$ we could have each $M_k(\mathbf{p}) = \max\{0, \lambda \mathcal{Z}_k(\mathbf{p})\}$; this will be useful later for obtaining convergence when we use this Gale-Nikaido transition function f to compute market equilibria.

It's also very important to remember that we're not making any assumption about what the "transition function" *really* is — *i.e.*, about how prices *really* move over time. We're merely *making up* a transition function f for which (a) f will have a fixed point, and (b) any fixed point of f will be an equilibrium.

Note that all three assumptions A1, A2, and A3 were used in the proof. A1 was used to ensure that f maps S into S, which might not be true if we replace S with a proper subset of S. A2 was used to establish that a fixed point of f is an equilibrium. And A3 was used to establish that f is continuous.

Some Issues

(1) If any individual's behavioral function is not single-valued — for example, if some consumer's indifference surfaces have flat segments or are not convex, or some firm has constant returns to scale — then the function $\mathcal{Z}(\cdot)$ also won't be single-valued. We'll have to alter our approach by using correspondences.

(2) If any consumer has strictly increasing preferences, then $\mathcal{Z}(\mathbf{p})$ won't be defined for some of the price-lists $\mathbf{p} \in S$ — those in which some prices are zero. We'll circumvent this problem by working with the disaggregated definition of Walrasian equilibrium, in which it's not assumed that the individuals necessarily have well-defined demands at each price-list.

A subsequent set of lecture notes addresses the existence-of-equilibrium question using correspondences and the disaggregated definition of Walrasian equilibrium. The disaggregated definition is the one we've used before, to study the relation of Walrasian equilibrium to the Pareto allocations and the core.

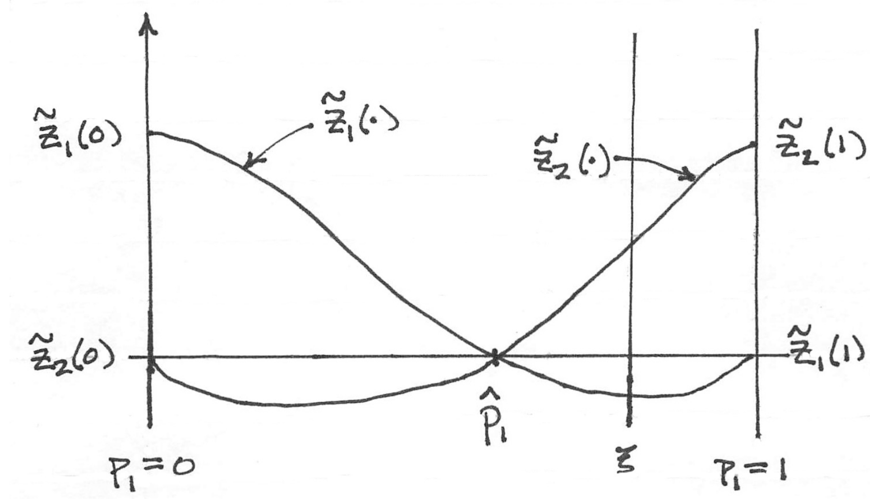

Figure 1

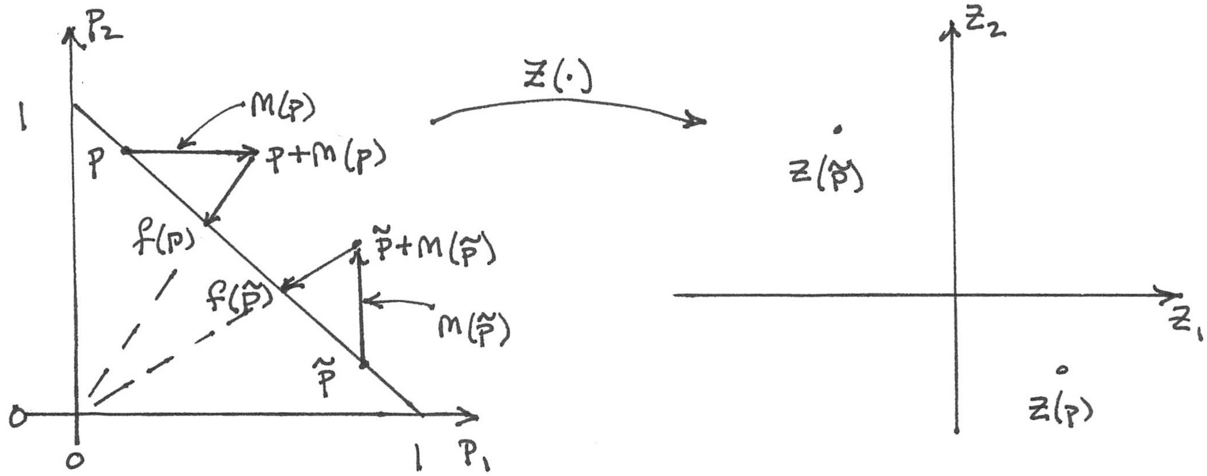

Figure 2